Notes on hydrostatic forces on plane surfaces

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for
CEE 3500 – *CEE Fluid Mechanics*

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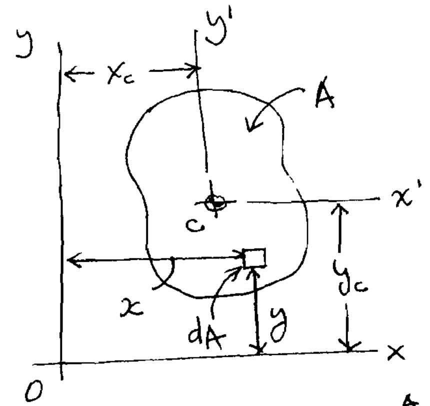
Area properties – show up in calculations related to forces on plane surfaces



First moment with respect to:

x-axis: $M_x^A = \int_A y\,dA = y_c A$

y-axis: $M_y^A = \int_A x\,dA = x_c A$

with area, $A = \int_A dA$, the coordinates of the centroid C

are: $x_c = \dfrac{M_y^A}{A} = \dfrac{\int x\,dA}{\int dA}$ and $y_c = \dfrac{M_x^A}{A} = \dfrac{\int y\,dA}{\int dA}$

Moments of inertia with respect to:

x-axis: $I_x = \int_A y^2\,dA$

y-axis: $I_y = \int_A x^2\,dA$

ORIGIN (polar moment of inertia):

$I_O = I_x + I_y = \int (x^2+y^2)\,dA = \int r^2\,dA$

with $r^2 = x^2+y^2$

Associated with the moments of inertia are the radii of gyration:

$k_x = \sqrt{\dfrac{I_x}{A}}$

$k_y = \sqrt{\dfrac{I_y}{A}}$

$k_O = \sqrt{\dfrac{I_O}{A}}$

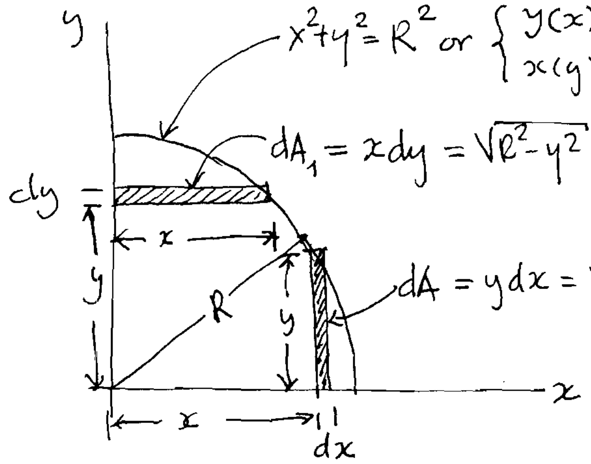
PARALLEL AXES THEOREM

$I_x = I_{x'} + y_c^2 A$, $I_O = I_{O'} + r_c^2 A$ with

$I_y = I_{y'} + x_c^2 A$ $\quad r_c^2 = x_c^2 + y_c^2$

primed quantities referred to centroidal axes x', y'

Calculations using elementary area strips



$x^2+y^2 = R^2$ or $\begin{cases} y(x) = \sqrt{R^2-x^2} \\ x(y) = \sqrt{R^2-y^2} \end{cases}$

$dA_1 = x\,dy = \sqrt{R^2-y^2}\,dy$

$dA = y\,dx = \sqrt{R^2-x^2}\,dx$

AREA: $A = \int dA_1$ or $\int dA_2$

$$A = \int_{y=0}^{y=R} \sqrt{R^2-y^2}\,dy = \int_{\theta=0}^{\theta=\pi/2} R\cos\theta \cdot R\cos\theta\,d\theta$$

change of variable

$y = R\sin\theta \Rightarrow \begin{cases} y=0 \to \theta = 0 \\ y = R \to \theta = \pi/2 \end{cases}$

$dy = R\cos\theta\,d\theta$

$\sqrt{R^2-y^2} = R\cos\theta$

$$A = R^2\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{R^2}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{R^2}{2}\left[\int_0^{\pi/2} d\theta + \int_0^{\pi/2}\cos 2\theta\,d\theta\right] = \frac{R^2}{2}\left[\theta\Big|_0^{\pi/2} + \left(\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}\right]$$

$$A = \frac{R^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}(0-0)\right] = \frac{\pi R^2}{4}$$ ← we could have found this out since it is 1/4 of the area of a circle (πR^2)

First moments of area

$$M_x^A = \int_A y\,dA_1 = \int_{y=0}^{y=R} y\sqrt{R^2-y^2}\,dy = \int_0^{\pi/2} R\sin\theta \cdot R\cos\theta \cdot R\cos\theta\,d\theta = R^3\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta = R^3\int_1^0 u^2(-du)$$

$$= -R^3\left.\frac{u^3}{3}\right|_1^0 = -\frac{R^3}{3}(0-1) = \frac{R^3}{3}$$

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CHANGE OF VARIABLE:

$u=\cos\theta \longrightarrow \begin{cases}\theta=0,\ u=1\\ \theta=\pi/2,\ u=0\end{cases}$

$du=-\sin\theta\,d\theta$

similarly, $M_y^A = \int_A x\,dA_2 = \int_{x=0}^{x=R} x\sqrt{R^2-x^2}\,dx = R^3/3$ (use $x=R\sin\theta$)

Thus, centroidal coordinates: $\boxed{x_c = \frac{M_y^A}{A} = \frac{R^3/3}{\pi R^2/4} = \frac{4R}{3\pi},\quad y_c = \frac{M_x^A}{A} = \frac{4R}{3\pi}}$

Moments of inertia

$$I_x = \int y^2 dA_1 = \int_{y=0}^{y=R} y^2\sqrt{R^2-y^2}\,dy = \int_{\theta=0}^{\theta=\pi/2} R^2\sin^2\theta \cdot R\cos\theta \cdot R\cos\theta\,d\theta = R^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

use $\left\{\begin{array}{l}\sin\theta\cos\theta = 1/2\sin 2\theta\\ \sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)\end{array}\right\} \Rightarrow I_x = \frac{R^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{R^4}{8}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta$

$$= \frac{R^4}{8}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\cos 4\theta\,d\theta\right] = \frac{R^4}{8}\left[\theta\Big|_0^{\pi/2} - \frac{1}{4}\sin(4\theta)\Big|_0^{\pi/2}\right]$$

$$\boxed{I_x = \frac{R^4}{8}\left[\frac{\pi}{2} - \frac{1}{4}(0-0)\right] = \frac{\pi R^4}{16} = \frac{\pi (D/2)^4}{16} = \frac{\pi D^4}{16\times 16} = \frac{\pi D^4}{256}}$$

R = radius

$D=2R$ = diameter

similarly $\boxed{I_y = \int y^2 dA_2 = \int_{x=0}^{x=R} x^2\sqrt{R^2-x^2}\,dx = \frac{\pi R^4}{16} = \frac{\pi D^4}{256}}$

radii of gyration

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\pi R^4/16}{\pi R^2/4}} = \sqrt{R^2\cdot\frac{4}{16}} = \sqrt{\frac{R^2}{4}} = \frac{R}{2}$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\pi R^4/16}{\pi R^2/4}} = \frac{R}{2}$$

$$A = \int_0^R \sqrt{R^2-y^2}\,dy \quad (1)$$

$$M_x^A = \int_0^R y\sqrt{R^2-y^2}\,dy$$

$$I_x = \int_0^R y^2\sqrt{R^2-y^2}\,dy$$

Calculating these integrals with the HP49G+

(1) ↱ $\xrightarrow{\text{EQW}}$ ↱ $\xrightarrow{\int}$ 0 ▷ ALPHA R ▷ √x ALPHA R y^x 2 △ △ − ALPHA ↰ Y y^x 2 ▷ ALPHA ↰ Y ENTER ENTER
EVAL ⟹ $\frac{R^2\cdot\pi}{4}$

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(2) ↱ $\xrightarrow{\text{EQW}}$ ↱ $\xrightarrow{\int}$ 0 ▷ ALPHA R ▷ ALPHA ↰ Y × √x ALPHA R y^x 2 △ △ − ALPHA ↰ Y y^x 2 ▷ ALPHA ↰ Y ENTER ENTER EVAL ⟹ $\frac{R^2|R|}{3} = \frac{R^3}{3}$

(3) ↱ $\xrightarrow{\text{EQW}}$ ↱ $\xrightarrow{\int}$ 0 ▷ ALPHA R ▷ ALPHA ↰ Y △ △ y^x 2 × √x ALPHA R y^x 2 △ △ − ALPHA ↰ Y y^x 2 ▷ ALPHA ↰ Y ENTER ENTER
EVAL ⟹ $\frac{R^4\cdot\pi}{16}$

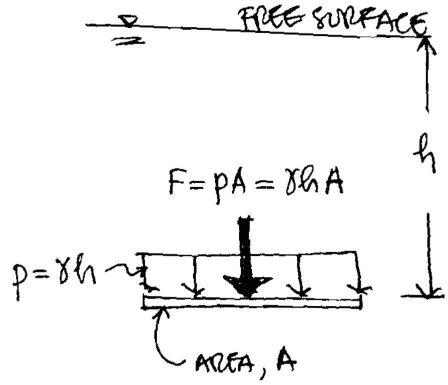
Calculating these integrals using the TI-89 Titanium

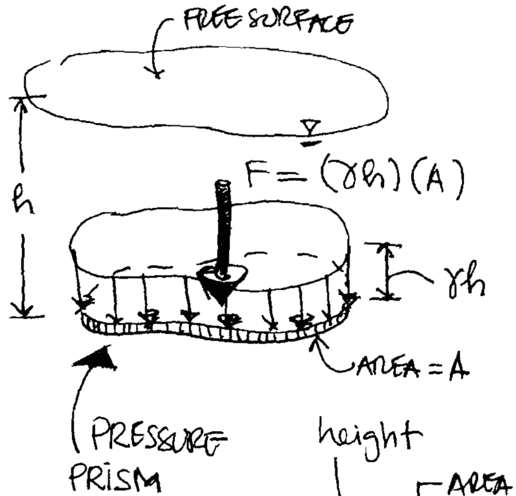
(1) 2nd $\xrightarrow{\int}$ 2nd $\xrightarrow{\sqrt{}}$ ALPHA ↑ $\xrightarrow{R}$ ^ 2 − Y ^ 2) , Y , 0 , ALPHA ↑ R) ENTER ⟹ $\frac{\pi\cdot r\cdot|r|}{4}$, should be $\frac{\pi R^2}{4}$

(2) 2nd $\xrightarrow{\int}$ Y × 2nd √ ALPHA ↑ $\xrightarrow{R}$ ^ 2 − Y ^ 2) , Y , 0 , ALPHA ↑ $\xrightarrow{R}$) Enter ⟹ $\frac{|r^3|}{3}$, should be $\frac{R^3}{3}$

(3) 2nd $\xrightarrow{\int}$ Y ^ 2 × 2nd $\xrightarrow{\sqrt{}}$ ALPHA ↑ $\xrightarrow{R}$ ^ 2 − Y ^ 2) , Y , 0 , ALPHA ↑ $\xrightarrow{R}$) Enter ⟹ $\frac{\pi\cdot r\cdot|r^3|}{16}$, should be $\frac{\pi R^4}{16}$

FORCE ON A HORIZONTAL PLANE AREA

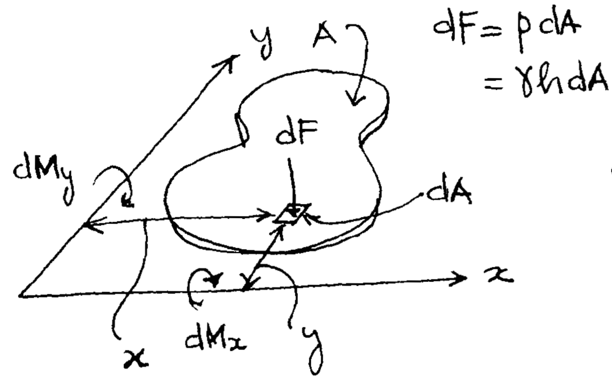


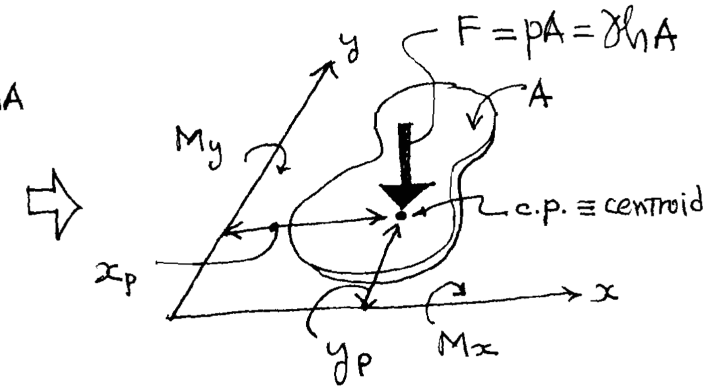


height ↓ AREA

NOTE: FORCE = volume of pressure prism, in this case $F = (\gamma h)(A)$

LOCATION OF FORCE ON A HORIZONTAL PLANE AREA (CENTER OF PRESSURE)





MOMENTS EXERTED BY FORCE dF:

- about x: $dM_x = y \cdot dF$
- about y: $dM_y = x \cdot dF$

MOMENT EXERTED by TOTAL FORCE F:

- about x: $M_x = y_p \cdot F$
- about y: $M_y = x_p \cdot F$

The total moments can be calculated as follows:

$$M_x = \iint_A dM_x = \iint_A y\,dF = \iint_A y\,p\,dA = \iint_A y\,\gamma h\,dA = \gamma h \iint_A y\,dA$$

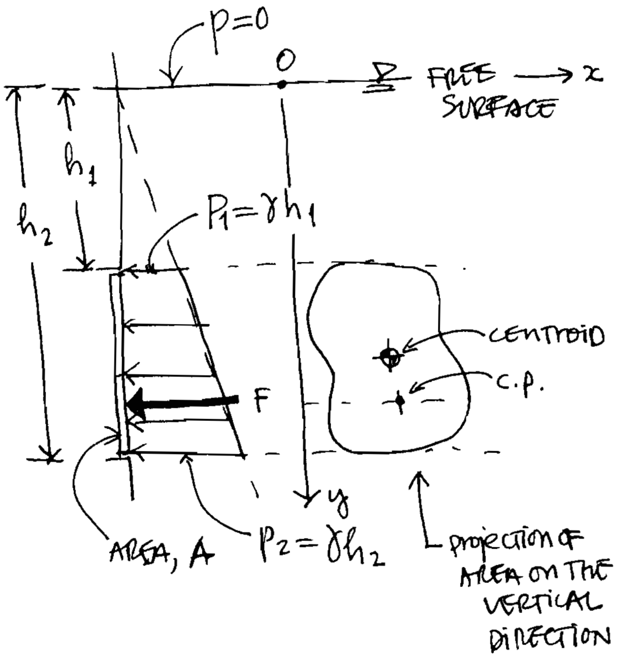
$$M_y = \iint_A dM_y = \iint_A x\,dF = \iint_A x\,p\,dA = \iint_A x\,\gamma h\,dA = \gamma h \iint_A x\,dA$$

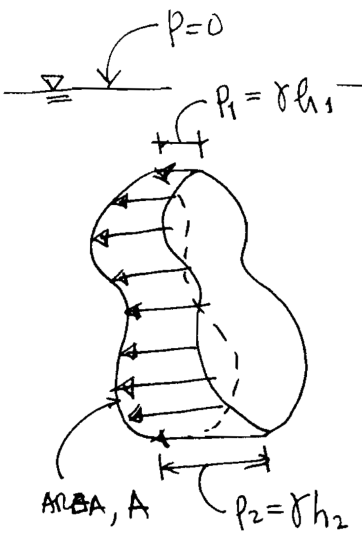
Thus, $y_p F = y_p \gamma h A = \gamma h \iint_A x\,dA \Rightarrow y_p = \frac{1}{A}\iint x\,dA$

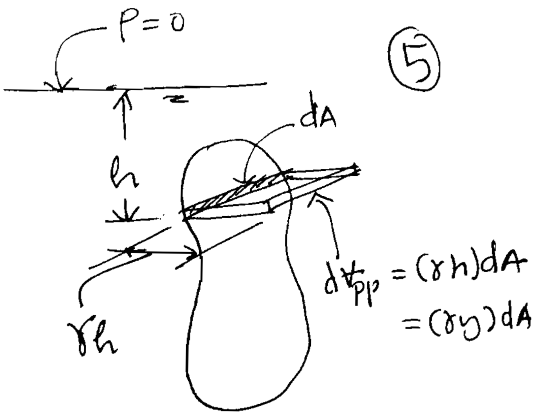
$x_p F = x_p \gamma h A = \gamma h \iint_A y\,dA \Rightarrow x_p = \frac{1}{A}\iint y\,dA$

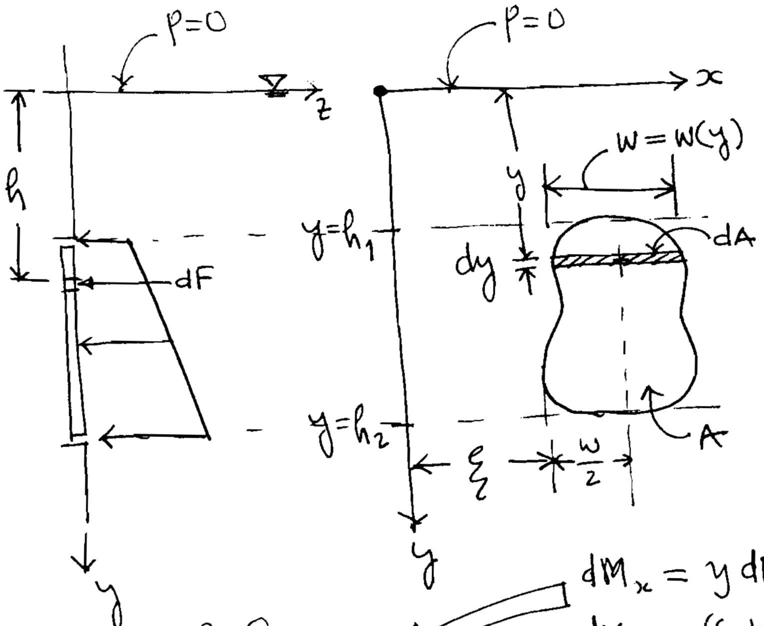
$\Big\}$ CENTROIDAL coordinates

FORCE ON A VERTICAL PLANE AREA









NOTE: $y = h$

AREA ELEMENT, $dA = w(y)\,dy$

PRESSURE ON dA, $p = \gamma h = \gamma y$

FORCE ON dA, $dF = p\,dA = \gamma y\, w(y)\,dy$

TOTAL FORCE ON A:

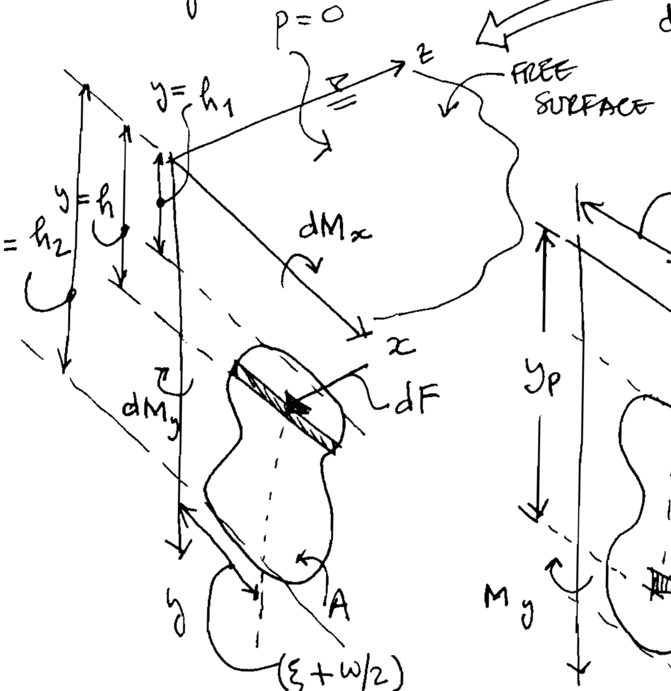
$$F = \int_{h_1}^{h_2} dF = \int_{h_1}^{h_2} \gamma y \underbrace{w(y)\,dy}_{dA}$$

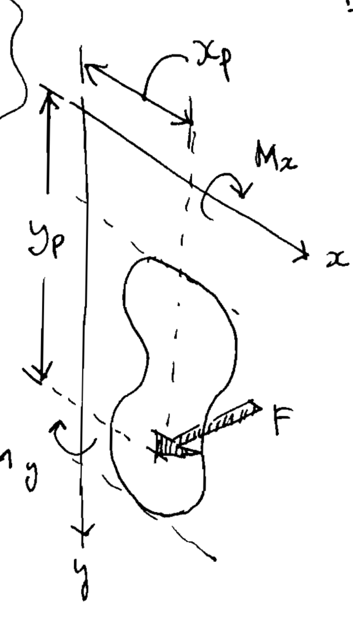
$$= \gamma \int_{h_1}^{h_2} y\,dA = \int_{h_1}^{h_2} \gamma y\,dA$$

$$= \int_{h_1}^{h_2} dV_{pp} = V_{pp} =$$ volume of pressure prism

$dM_x = y\,dF$

$dM_y = (\xi + \frac{w}{2})\,dF$





MOMENTS:

$$M_x = y_p \cdot F = \int dM_x = \int_{h_1}^{h_2} y\,dF$$

$$M_y = x_p \cdot F = \int dM_y = \int_{h_1}^{h_2} (\xi + \frac{w}{2})\,dF$$

$$M_x = \int_{h_1}^{h_2} y\,dF = \int_{h_1}^{h_2} y \cdot \gamma y\, w(y)\,dy$$

$$= \int_{h_1}^{h_2} \gamma y^2\,dA = y_p \cdot \int_{h_1}^{h_2} \gamma y\,dA$$

(continues in next page) ⇒

since γ = constant $\Rightarrow$ $\cancel{\gamma}\underbrace{\int_{h_1}^{h_2} y^2 dA}_{\text{moment of inertia } (I_x) \text{ w.r.t. } x} = \cancel{\gamma} y_p \underbrace{\int_{h_1}^{h_2} y\, dA}_{\text{first moment w.r.t. } x \;=\; y_c \cdot A}$ $\Rightarrow$ $I_x = y_c \cdot A \cdot y_p$ $\Rightarrow$ $\boxed{y_p = \frac{I_x}{y_c A}}$

A = area
y_c = y-coordinate for centroid
I_x = moment of inertia w.r.t. to x axis.

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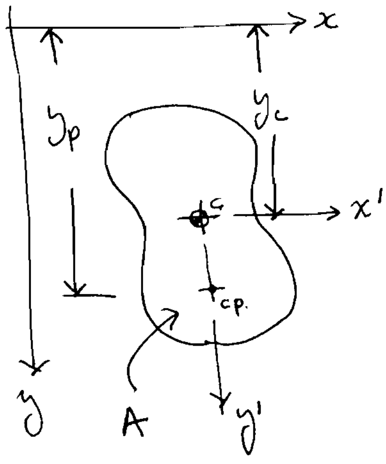
PARALLEL-AXES THEOREM



Parallel axes theorem indicates that $I_x = I_{x'} + A y_c^2$

$I_{x'}$: centroidal moment of inertia w.r.t. to x; y_c: distance between axes

Thus, $y_p = \frac{I_x}{y_c A} = \frac{I_{x'} + A y_c^2}{y_c A}$

$$\boxed{y_p = y_c + \frac{I_{x'}}{y_c A}}$$

NOTE: since $I_{x'} > 0$, $y_c > 0$, $A > 0$ $\Rightarrow$ $y_p > y_c$, i.e., the center of pressure is located below the centroid of the area

From $M_y = x_p \cdot F = \int_{h_1}^{h_2} (\xi + w/2) dF$ $\Rightarrow$ $x_p \cdot \int_{h_1}^{h_2} \gamma y w(y) dy = \int_{h_1}^{h_2} (\xi + \frac{w}{2}) \gamma y w(y) dy$

$\Rightarrow x_p \cdot \cancel{\gamma} \underbrace{\int_{h_1}^{h_2} y \underbrace{w(y) dy}_{dA}} = \cancel{\gamma}\int_{h_1}^{h_2} \xi y w dy + \cancel{\gamma}\int_{h_1}^{h_2} y \frac{w^2}{2} dy$

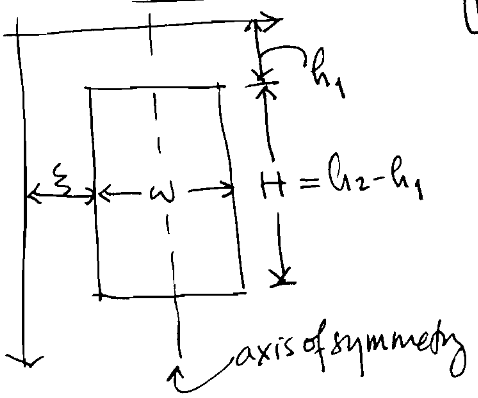
$$\boxed{x_p \cdot y_c \cdot A = \int_{h_1}^{h_2} (\xi + \frac{w}{2}) dA}$$

$\rightarrow$ specific case: w = constant and ξ = constant

$\int_{h_1}^{h_2} (\xi + \frac{w}{2}) y w\, dy = (\xi + \frac{w}{2}) w \int_{h_1}^{h_2} y\, dy = (\xi + \frac{w}{2}) \frac{w}{2} (h_2^2 - h_1^2)$

$= w(\xi + \frac{w}{2}) H (h_2 + h_1)$

$h_2^2 - h_1^2 = (h_2 - h_1)(h_2 + h_1) = H(h_2 + h_1)$



also, $y_c = h_1 + H/2$

$x_p (h_1 + H/2) \cancel{w} \cancel{H} = \cancel{w} (\xi + \frac{w}{2}) \frac{\cancel{H}}{2} (h_2 + h_1)$

$x_p = \frac{\xi + w/2}{h_1 + H/2} \cdot (h_2 + h_1) \cdot \frac{1}{2} = \frac{\xi + w/2}{\frac{h_1 + h_2}{2}} \cdot \frac{h_1 + h_2}{2} = \xi + \frac{w}{2}$

$h_1 + \frac{H}{2} = h_1 + \frac{h_2 - h_1}{2} = \frac{2h_1}{2} + \frac{h_2}{2} - \frac{h_1}{2}$
$= \frac{h_1 + h_2}{2} = y_c$

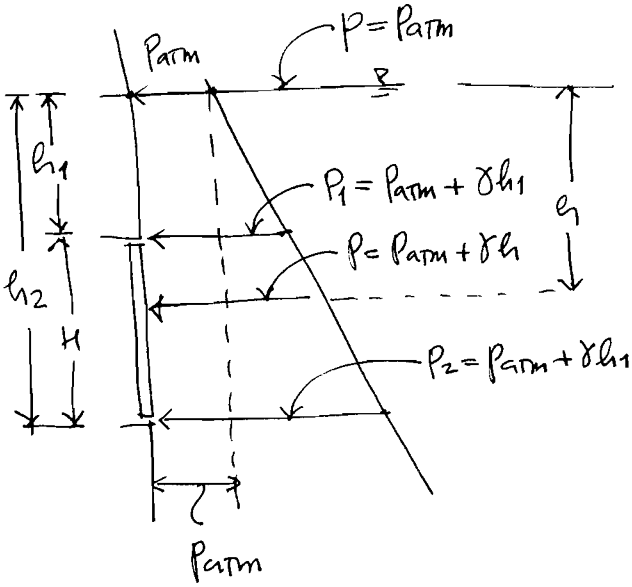
$\Rightarrow$ $x_p = x_c$

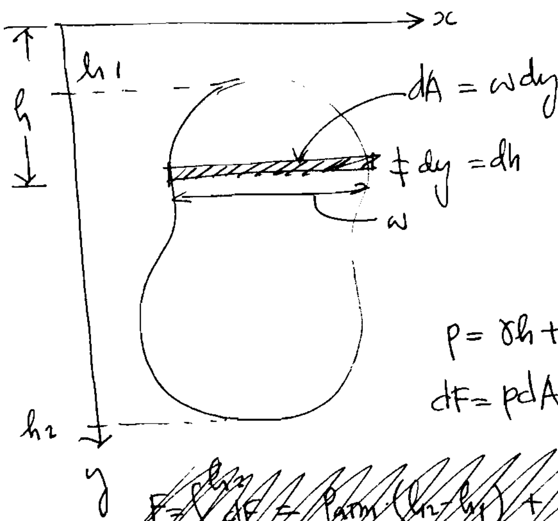
similar result for symmetric shapes

NOTE: For symmetric shapes, select y as the axis of symmetry $\Rightarrow$ $x_p = x_c = 0$

NOTE: $\boxed{F = \gamma \int_{h_1}^{h_2} y\, dA = \gamma y_c A = \gamma h_c A = p_c A \equiv (\text{pressure at centroid}) \times (\text{area})}$

NOTE: we used gage pressure in the previous example. What if we use absolute pressure?





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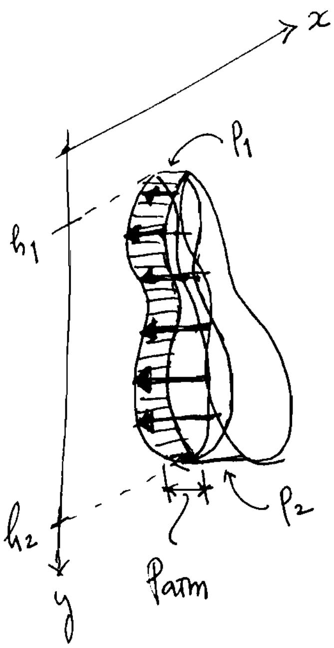
$$p = \gamma h + P_{atm}$$

$$dF = p\,dA = (P_{atm} + \gamma h)\,dA$$

$$\cancel{F = \int_{h_1}^{h_2} dF = P_{atm}(h_2 - h_1) + \int_{h_1}^{h_2} \gamma h\,dA}$$

$$\cancel{= P_{atm} H + \gamma h_c A}$$



FORCE

$$F = \int dF = \int_{h_1}^{h_2} P_{atm}\,dA + \int_{h_1}^{h_2} \gamma h\,dA = P_{atm}\int_{h_1}^{h_2} dA + \int_{h_1}^{h_2} \gamma h\,dA$$

$$\boxed{F = P_{atm} \cdot A + \gamma h_c A = (P_{atm} + \gamma h_c)A = P_c \cdot A}$$

MOMENT WITH RESPECT TO X-AXIS

$$y_p \cdot F = \int_{h_1}^{h_2} y\,p\,dA$$

$$y_p (P_{atm} + \gamma h_c) A = \int_{h_1}^{h_2} y\,P_{atm}\,dA + \int_{h_1}^{h_2} \gamma y^2\,dA$$

$$y_p A P_{atm} + y_p \gamma h_c A = P_{atm} y_c A + \gamma I_x \,, \quad I_x = I_{x'} + A y_c^2$$

$$y_p A P_{atm} + y_p \gamma h_c A = P_{atm} y_c A + \gamma I_{x'} + \gamma A y_c^2$$

$$y_p A (P_{atm} + \gamma h_c) = P_{atm} y_c A + \gamma A y_c^2 + \gamma I_{x'}$$

$$y_p A (P_{atm} + \gamma h_c) = (P_{atm} + \gamma y_c) A y_c + \gamma I_{x'}$$

$$\boxed{y_p = y_c + \frac{\gamma I_{x'}}{(P_{atm} + \gamma h_c)A} = y_c + \frac{\gamma I_{x'}}{P_c A}}$$

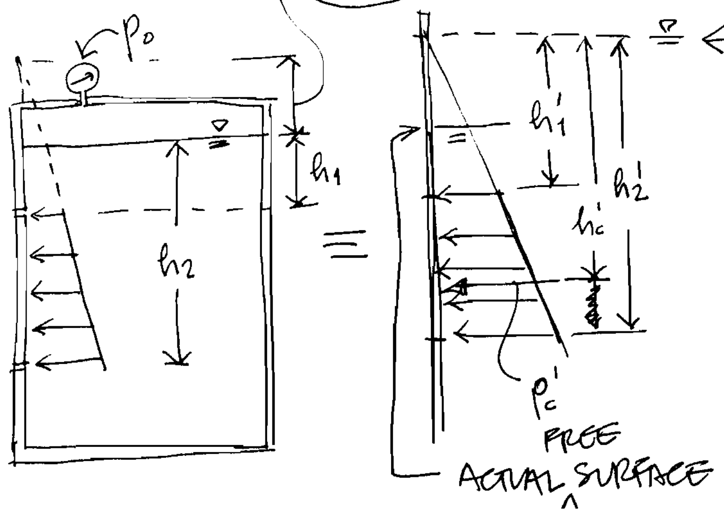
NOTE: these calculations apply also to gage pressure calculations when the free surface pressure is NOT ZERO $\longrightarrow$ pressurized vessel

VIRTUAL FREE-SURFACE

$h_0 = P_0/\gamma$

$$\left.\begin{aligned} h_1' &= h_1 + h_0 = h_1 + \frac{P_0}{\gamma} \\ h_2' &= h_2 + h_0 = h_2 + \frac{P_0}{\gamma} \end{aligned}\right\} \leftarrow$$ use depths h_1', h_2', etc. measured from the virtual free surface

(8)



VIRTUAL FREE SURFACE of zero pressure located at $h_0 = P_0/\gamma$ above (or below, if $P_0 < 0$) of actual free surface

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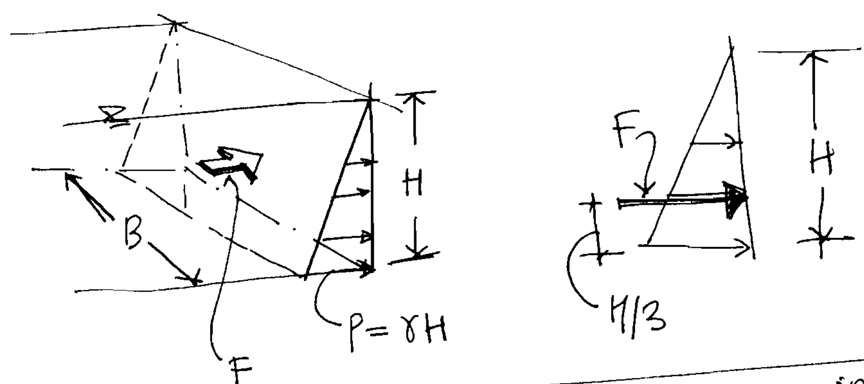
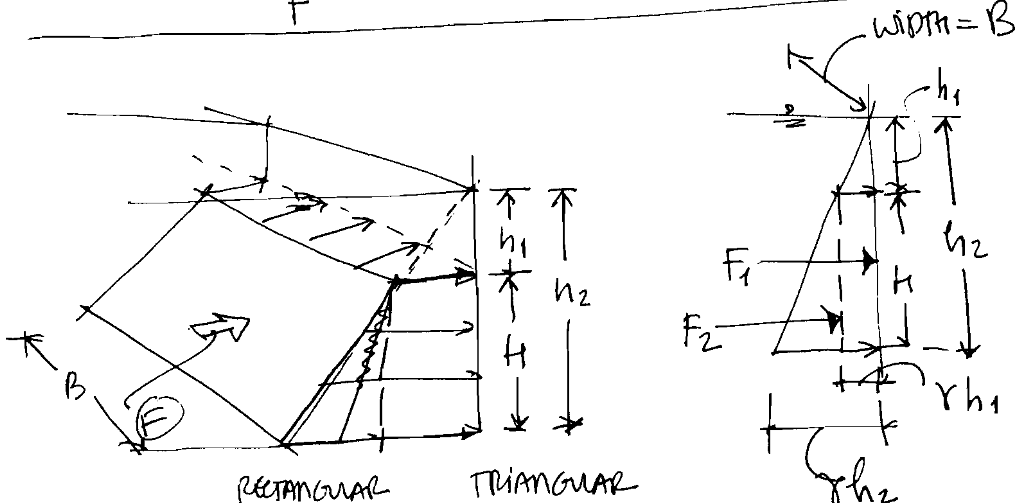
FORCE = VOLUME OF PRISM OF PRESSURES, $F = \frac{1}{2}(\gamma H)(H)B = \frac{1}{2}\gamma B H^2$





RECTANGULAR CONTRIBUTION

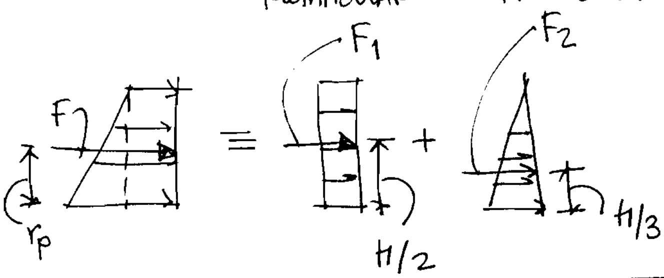
$F_1 = (\gamma h_1)(H)(B) = \gamma h_1 H B$

TRIANGULAR CONTRIBUTION

$F_2 = \frac{1}{2}\gamma(h_2 - h_1)(H)(B)$
$= \frac{1}{2}\gamma H^2 B$

$F = F_1 + F_2 = \gamma h_1 H B + 1/2\, \gamma H^2 B$

$$\boxed{F = \gamma H B\left(h_1 + \frac{H}{2}\right)}$$

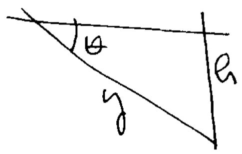


MOMENTS: $r_p \cdot F = \frac{H}{2} \cdot F_1 + \frac{H}{3} \cdot F_2$

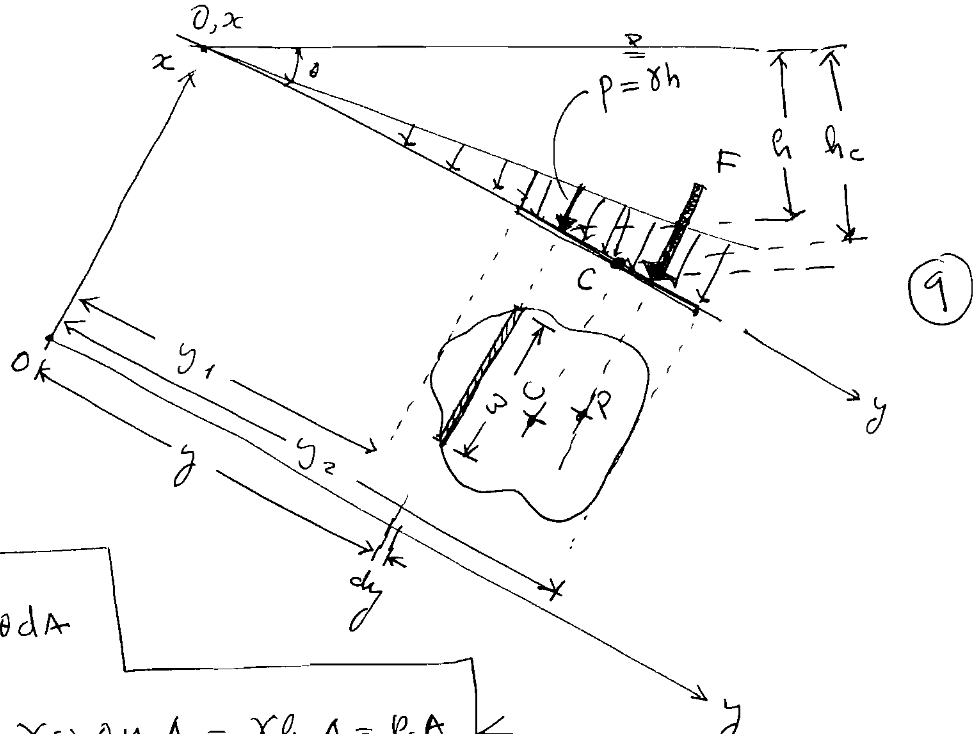
$r_p \cdot \left(\gamma H B \cdot (h_1 + H/2)\right) = \frac{H}{2} \cdot \gamma h_1 H B + \frac{H}{3} \cdot \frac{1}{2}\gamma H^2 B$

$\cancel{\gamma H B}\left(h_1 + \frac{H}{2}\right) r_p = \frac{1}{2}\cancel{\gamma} H^2 \cancel{B}\left(h_1 + \frac{H}{3}\right) \Rightarrow \boxed{r_p = \frac{H}{2} \cdot \frac{h_1 + H/3}{h_1 + H/2}}$

Force on a surface at an angle θ with respect to the free surface



$$\boxed{h = y\sin\theta}$$



$$dF = p\,dA = \gamma h\,dA$$
$$= \gamma y \sin\theta\, dA$$

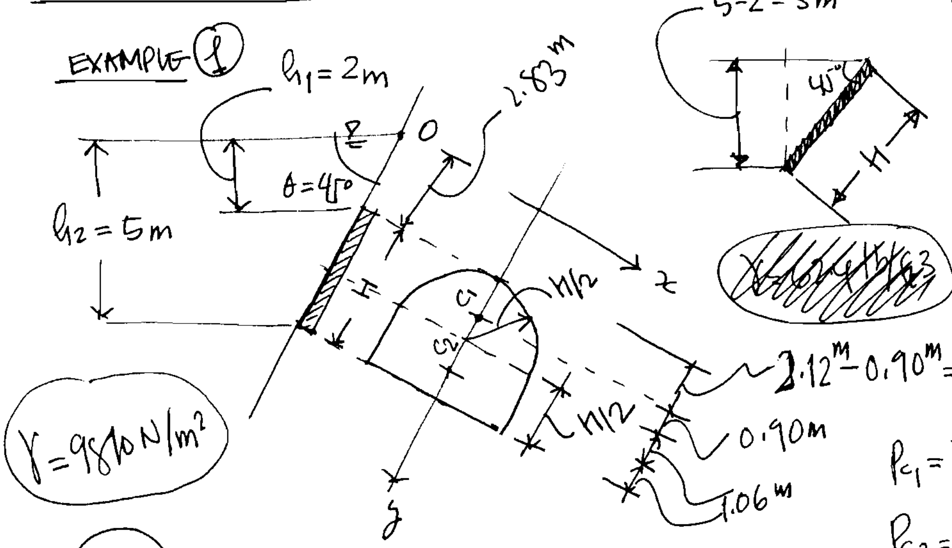
$$\boxed{F = \int_{y_1}^{y_2} dF = \int_{y_1}^{y_2} \gamma y \sin\theta\, dA = \gamma\sin\theta \int_{y_1}^{y_2} y\,dA = \gamma\sin\theta\, y_c A = \gamma h_c A = p_c A}$$ ← volume of pressure prism

$$M_x = y_p \cdot F = \int_{y_1}^{y_2} y\,dF \Rightarrow y_p \cdot \gamma \sin\theta\, y_c A = \int_{y_1}^{y_2} \gamma y^2 \sin\theta\, dA$$

$$y_p \cdot \cancel{\gamma\sin\theta}\, y_c A = \cancel{\gamma\sin\theta} \int_{y_1}^{y_2} y^2\,dA \Rightarrow y_p y_c A = I_x \Rightarrow y_p = \frac{I_x}{y_c A}$$

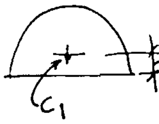
with $I_x = I_{x'} + y_c^2 A$, $\boxed{y_p = y_c + \frac{I_{x'}}{y_c A}}$

EXAMPLE 1

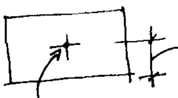


$$\sin 45° = \frac{3^m}{H} \Rightarrow H = \frac{3^m}{\sin 45°} = 4.24^m$$
$$H/2 = 2.12^m$$

$$h_{c_1} = 2^m + 1.22^m \sin 45° = 2.86^m$$
$$h_{c_2} = 2^m + (2.12^m + 1.06^m)\sin 45° = 4.25^m$$



$$\frac{4r}{3\pi} = \frac{4}{3\pi}\left(\frac{H}{2}\right) = \frac{2H}{3\pi} = \frac{2\times 4.24^m}{3\pi} = 0.90^m$$



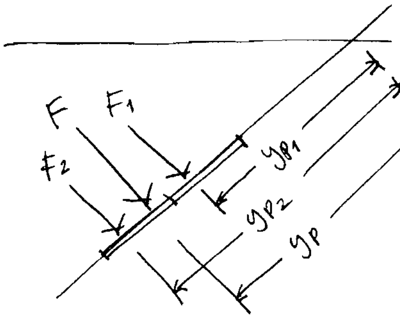
$$\frac{H/2}{2} = \frac{H}{4} = \frac{4.24^m}{4} = 1.06^m$$

$$A_1 = \frac{1}{2}\pi\left(\frac{H}{2}\right)^2 = \frac{\pi}{8}H^2 = \frac{\pi}{8}\times(4.24m)^2 = 7.06m^2$$
$$A_2 = H(H/2) = H^2/2 = \frac{1}{2}\times(4.24m)^2 = 9.00\,m^2$$

$$p_{c_1} = \gamma h_{c_1} = \left(9810\frac{N}{m^3}\right)(2.86^m) = 28056.6\ N/m^2$$
$$p_{c_2} = \gamma h_{c_2} = \left(9810\frac{N}{m^3}\right)(4.25^m) = 41692.5\ N/m^2$$

$$F_1 = p_1 A_1 = 28056.6\frac{N}{m^2}\times 7.06m^2 = 198.08\ kN$$
$$F_2 = p_2 A_2 = 41692.5\frac{N}{m^2}\times 9.00m^2 = 375.23\ kN$$

$$F = F_1 + F_2 = 573.31 N$$



$$(I_o)_1 = \frac{\pi D^4}{128} = \frac{\pi \times (H/2)^4}{128} = \frac{\pi H^4}{2048} = \frac{\pi \times 4.24^4}{2048} = 0.50\ m^4$$

$$(I_o)_2 = \frac{bh^3}{12} = \frac{H(H/2)^3}{12} = \frac{H^4}{96} = \frac{4.24^4}{96} = 3.37\ m^4$$

$$y_{p_1} = y_{c_1} + \frac{(I_o)_1}{y_{c_1}A_1} = 4.04^m + \frac{0.50\ m^4}{4.04^m \times 7.06\ m^2} = 4.06\ m$$

$F_1 = 198.08$ KN, $y_{p_1} = 4.06$ m
$F_2 = 375.23$ KN, $y_{p2} = 6.07$ m
$F = 573.13$ KN, $y_p = 5.37$ m

$$y_{p_2} = y_{c_2} + \frac{(I_o)_2}{y_{c_2}A_2} = 6.01^m + \frac{3.37\ m^4}{6.01^m \times 9.00\ m^2} = 6.07^m$$

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$$y_{c_1} = \frac{h_{c_1}}{\sin 45^\circ} = \frac{2.86^m}{\sin 45^\circ} = 4.04^m$$

$$y_{c_2} = \frac{h_{c_2}}{\sin 45^\circ} = \frac{4.25^m}{\sin 45^\circ} = 6.01^m$$

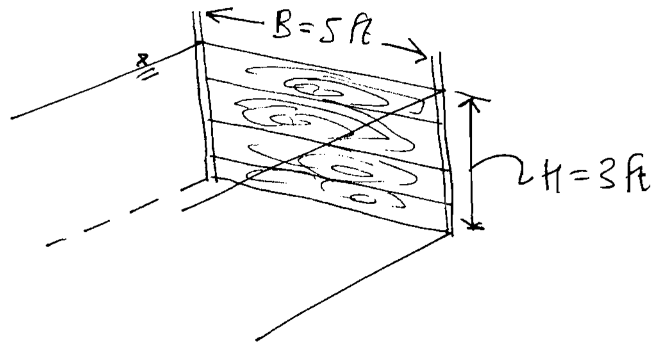
$$y_p \cdot (F_1 + F_2) = y_{p_1}F_1 + y_{p_2}F_2$$

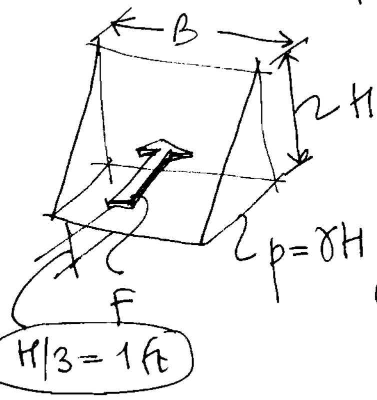
$$y_p \cdot (573.13\ KN) = 4.06^m \times 198.08\ KN + 6.07^m \times 375.23\ KN$$

$$y_p = 5.37^m$$

EXAMPLE 2

FIND THE FORCE ON THE GATE





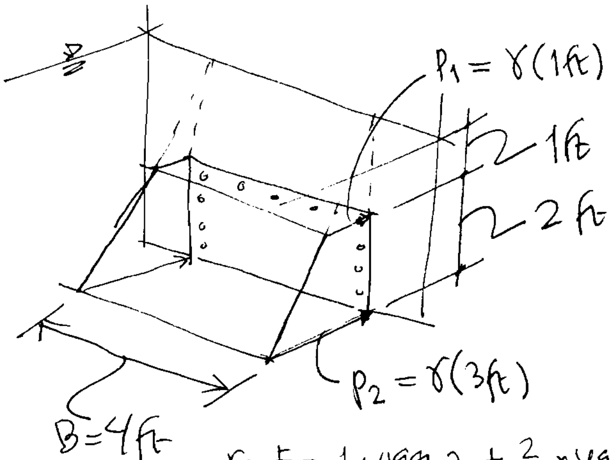
$$F = \tfrac{1}{2}(\gamma H)(H)B = \tfrac{1}{2}\gamma B H^2$$

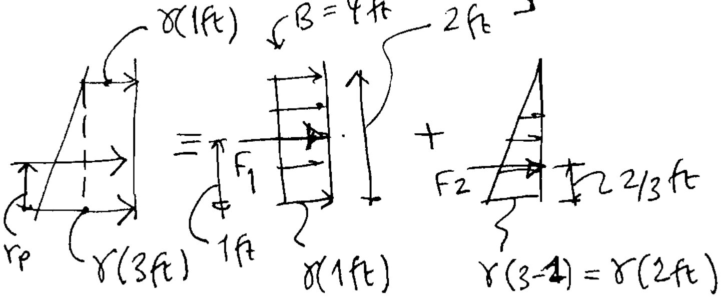
$$\gamma = 62.4\ \frac{lb}{ft^3}$$

$$F = \tfrac{1}{2}\left(62.4\ \frac{lb}{ft^3}\right)(5\ ft)(3\ ft)^2$$

$$F = 1404\ lb$$

EXAMPLE 3 – FIND FORCE ON GATE





$$F_1 = \gamma(1ft)(2ft)(4ft) = \gamma(8ft^3) = 62.4\ lb/ft^3 \times 8ft^3 = 499.2\ lb$$

$$F_2 = \tfrac{1}{2}\gamma(2ft)(2ft)(4ft) = \gamma(8ft^3) = 499.2\ lb$$

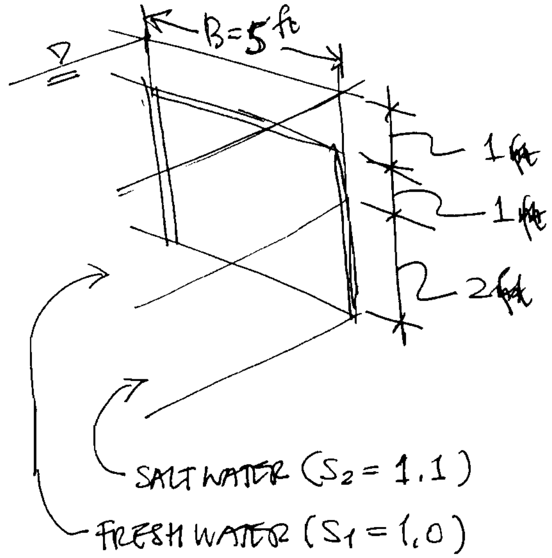
$$\boxed{F = F_1 + F_2 = 2 \times 499.2\ lb = 998.4\ lb}$$

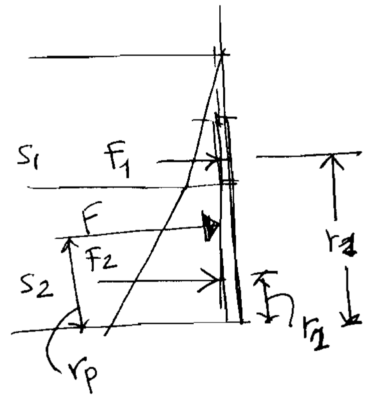
$$r_p \cdot F = 1 \times 499.2 + \tfrac{2}{3} \times 499.2$$

$$r_p \cdot 998.4 = 832.00$$

$$r_p = \frac{832.00}{998.40} = 0.833\ ft$$

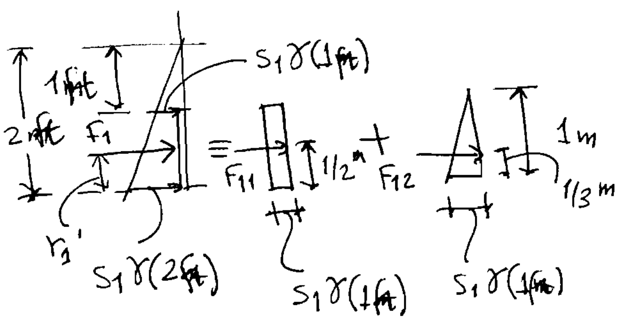
EXAMPLE 4





$\gamma = 62.4\ lb/ft^3$

FOR F_1



$F_{11} = s_1\gamma(1ft)(1ft)(5m) = 5ft^3 \cdot s_1\gamma$

$F_{12} = \frac{1}{2}s_1\gamma(1ft)(1ft)(5ft) = 2.5ft^3 s_1\gamma$

$F_1 = F_{11} + F_{12} = 7.5ft^3 \cdot s_1 \cdot \gamma$

$r_1' \times 7.5ft^3 \cdot \cancel{s_1\gamma} = \frac{1}{2}ft \times 5ft^3 \cancel{s_1\gamma} + \frac{1}{3}ft \times 2.5ft^3 \cancel{s_1\gamma}$

$r_1' = 0.44\ ft$

$r_1 = r_1' + 2ft = 2.44\ ft$

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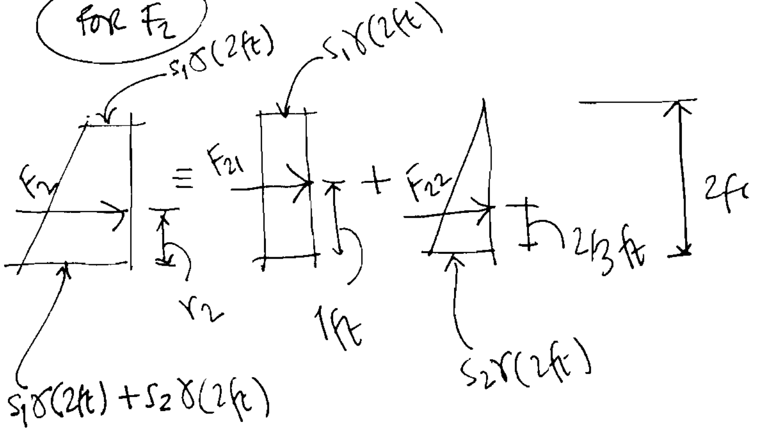
FOR F_2



$F_{21} = s_1\gamma(2ft)(2ft)(5ft) = 20\ ft^3 s_1\gamma$

$F_{22} = \frac{1}{2}\cdot s_2 \cdot \gamma \cdot (2ft)(2ft)(5ft) = 10\ ft^3 s_2\gamma$

$F_2 = F_{21} + F_{22} = \gamma(20 s_1 + 10 s_2)$

$r_2 \cdot \cancel{\gamma}(20s_1 + 10s_2) = (1ft)(20ft^3 s_1\cancel{\gamma}) + (\frac{2}{3}ft)(10ft^3 s_2\cancel{\gamma})$

$r_2(20s_1 + 10s_2) = 20s_1 + \frac{20}{3}s_2$

$r_2 = \frac{20s_1 + 20/3 s_2}{20s_1 + 10s_2} = \frac{20\times1 + (20/3)\times1.1}{20\times1 + 10\times1.1} = 0.88\ ft$

TOTAL FORCE

$F_1 = 7.5 \times 1.0 \times 62.4 = 468\ lb$

$F_2 = 62.4 \times (20\times1.0 + 10\times1.1) = 1934.4\ lb$

$F = F_1 + F_2 = 2402.4\ lb$

MOMENTS

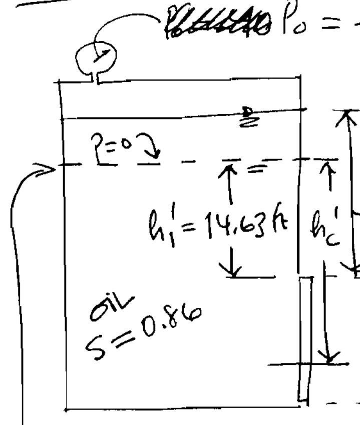
$r_p \cdot F = r_1F_1 + r_2F_2$

$r_p \cdot 2402.4 = 2.44 \times 468 + 0.88 \times 1934.4$

$r_p = 1.18\ ft$

EXAMPLE 5

$p_o = -20$ psi



(BELOW ACTUAL SURFACE SINCE $p_o < 0$)

with $s = 0.86$, $\gamma = 0.86 \times 62.4 \text{ lb/ft}^3 = 53.66 \text{ lb/ft}^3$

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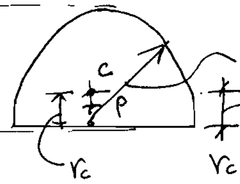
$$h_o = \frac{p_o}{\gamma} = \frac{(-20 \text{ lb/in}^2)(144 \text{ in}^2/\text{ft}^2)}{53.66 \text{ lb/ft}^3} = -5.36 \text{ ft}$$

$$h_c' = h_1' + (r - r_c) = 14.63 \text{ ft} + (10 \text{ ft} - 4.24 \text{ ft}) = 20.39 \text{ ft}$$

CENTROIDAL DEPTH



$$r_c = \frac{4r}{3\pi} = \frac{4 \times 10 \text{ ft}}{3\pi} = 4.24 \text{ ft}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\cdot\pi\cdot 10^2 = 157.08 \text{ ft}^2$$

$$I_o = \frac{\pi D^4}{128} = \frac{\pi \times 20^4}{128} = 3926.99 \text{ ft}^4$$

$$p_c' = \gamma h_c' = \left(53.66 \tfrac{\text{lb}}{\text{ft}^3}\right)(20.39 \text{ ft})$$

$$p_c' = 1094.12 \text{ lb/ft}^2$$

$$F = p_c' \cdot A = \left(1094.12 \tfrac{\text{lb}}{\text{ft}^2}\right)(157.08 \text{ ft}^2)$$

~~F = 171.87 k~~

$$\boxed{F = 171865.53 \text{ lb}}$$

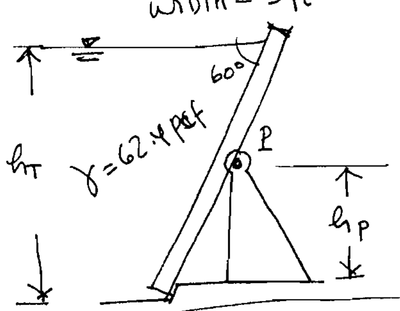
$y_c' = h_c' = 20.39$ ft

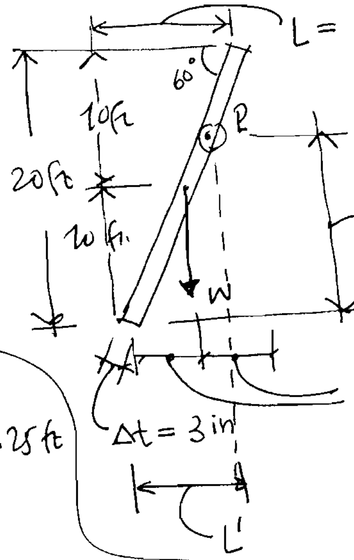
$$y_p' = y_c' + \frac{I_o}{y_c' A} = 20.39 \text{ ft} + \frac{3926.99 \text{ ft}^4}{(20.39 \text{ ft})(157.08 \text{ ft}^2)} = 21.61 \text{ ft}$$

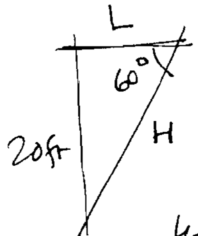
or $21.61 + 5.36 = 26.98$ ft from actual free surface

EXAMPLE 6

Given $h_T = 20$ ft, $h_p = ?$ if flashboard is about to tip?







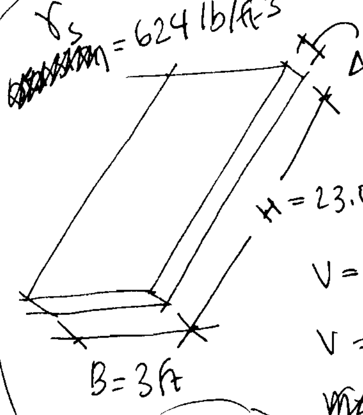
$$\sin 60° = \frac{20 \text{ ft}}{H}$$

$$H = \frac{20 \text{ ft}}{\sin 60°} =$$

$$H = 23.09 \text{ ft}$$

$$\tan 60° = \frac{20 \text{ ft}}{L}$$

$$L = \frac{20 \text{ ft}}{\tan 60°} = 11.55 \text{ ft}$$



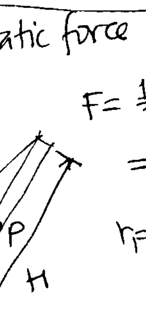
$V = HB\Delta t = 3 \times 23.09 \times 0.25$

$V = 17.32 \text{ ft}^3$

WEIGHT CALCULATION

$W = \gamma_s V = 624 \times 17.32$

$W = 10807.99$ lb



$$\tan 60° = \frac{h_p}{L'} \Rightarrow L' = \frac{h_p}{\tan 60°}$$

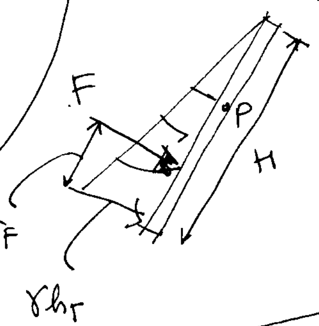
$$\sin 60° = \frac{h_p}{H'} \Rightarrow H' = \frac{h_p}{\sin 60°}$$

hydrostatic force

$$F = \tfrac{1}{2}\gamma h_T H B = \tfrac{1}{2}(62.4)(20)(23.09)(3)$$

$$= 43224.48 \text{ lb}$$

$$r_F = 1/3 H = \tfrac{1}{3} \times 23.09 = 7.7 \text{ ft}$$

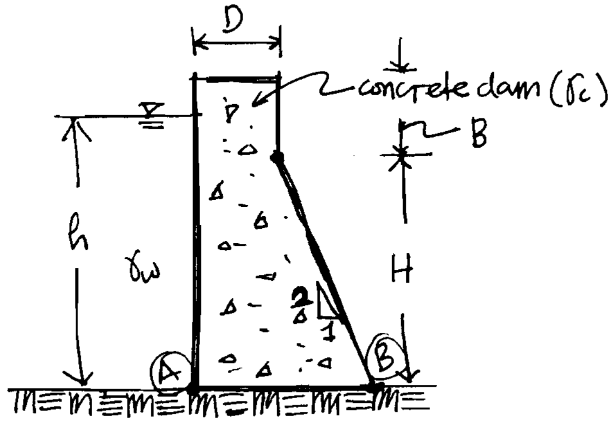


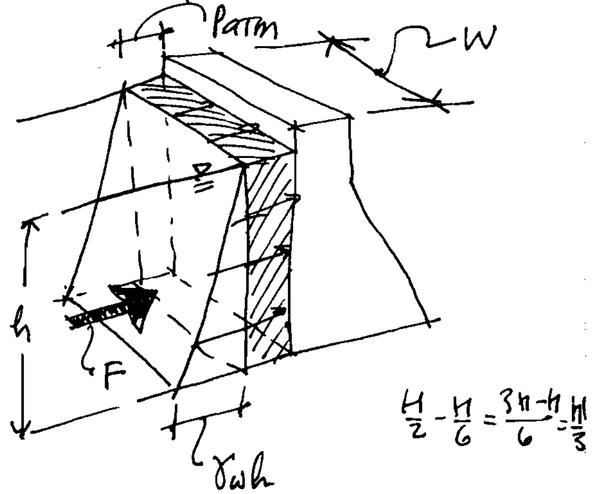
$$\circlearrowleft + \sum M_P = 0, \quad W(L' - 5.77) + F \times (H' - r_F) = 0$$

$$10807.99 \times \left(\frac{h_p}{\tan 60°} - 5.77\right) + 43224.48 \times \left(\frac{h_p}{\sin 60°} - 7.7\right) = 0 \Rightarrow \text{solve for } \boxed{h_p = 7.04 \text{ ft}}$$

$$6240 h_p - 62362.10 + 49910.77 h_p - 332828.5 = 0 \Rightarrow 56150.77 h_p = 395190.6$$

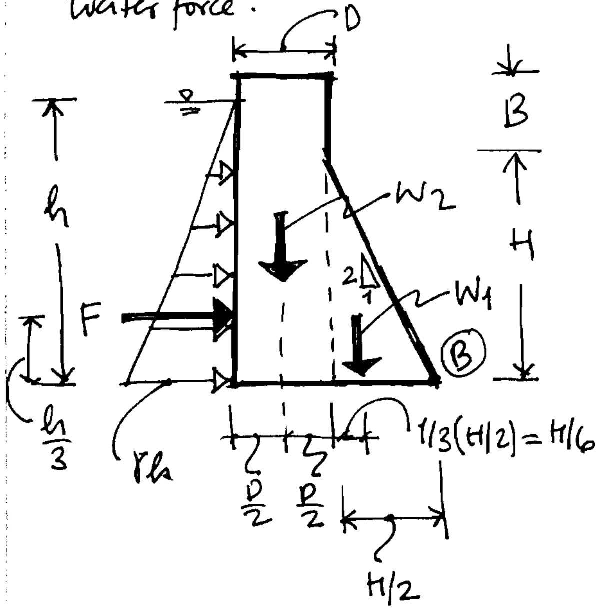
STABILITY OF A DAM: The figure shows a concrete dam (specific weight, γ_c) holding water at a depth h. Determine the height h so that the dam will tip at point B. Perform the analysis on a width w of the dam.





$\frac{H}{2} - \frac{H}{6} = \frac{3H - H}{6} = \frac{H}{3}$

We're tempted to add Patm in calculating the force from the water on the wall. However, Patm also acts on the other side of the wall, thus, using gage pressure is enough to calculate the water force.



$$F = \tfrac{1}{2}(\gamma_w h)(h)w = \tfrac{1}{2}\gamma_w h^2 w$$

$$W_1 = \tfrac{1}{2}(H)\left(\tfrac{H}{2}\right)(w)\gamma_c = \tfrac{1}{4}\gamma_c H^2 w$$

$$W_2 = D \times (B+H) \times w \times \gamma_c = \gamma_c D w (B+H)$$

AT TILTING POINT $\circlearrowleft\!+ \sum M_B = 0$

$$-F \times \frac{h}{3} + W_1 \times \left(\frac{H}{2} - \frac{H}{6}\right) + W_2 \times \left(\frac{H}{2} + \frac{D}{2}\right) = 0$$

$$-\tfrac{1}{2}\gamma_w h^2 w \frac{h}{3} + \tfrac{1}{4}\gamma_c H^2 w \frac{H}{3} + \gamma_c D w (B+H)\left(\frac{H+D}{2}\right) = 0$$

$$\tfrac{1}{6}\gamma_w h^3 \cancel{w} = \gamma_c \cancel{w}\left(\frac{1}{12}H^3 + \frac{1}{2}D(B+H)(H+D)\right)$$

$$h^3 = \frac{\gamma_c}{\gamma_w}\left(\frac{H^3}{2} + 3D(B+H)(H+D)\right)$$

$$\boxed{h = \sqrt[3]{\frac{\gamma_c}{\gamma_w}\left(\frac{H^3}{2} + 3D(B+H)(H+D)\right)}}$$